Comparative statics in symmetric, concave games

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Abstract

A concave game is one in which Kakutani's theorem implies the existence of a pure-strategy equilibrium, by a classic argument. This paper establishes an equilibrium comparative statics result for symmetric, concave games with one-dimensional action spaces. This result mirrors the well-known comparative statics results for supermodular games.

Keywords: Equilibrium comparative statics, concave games, Kakutani's theorem

1 Introduction

Games with strategic complementarities (e.g., supermodular games) are amenable to equilibrium comparative statics analysis, as is well known. This paper establishes that another class of games is similarly amenable to equilibrium comparative statics analysis: that is the class of symmetric, concave games with one-dimensional action spaces. This class of games is commonly encountered in applied game theory.

In concave games, Kakutani's theorem implies the existence of an equilibrium in pure strategies, by a classic argument. (A *concave game* is one where each players's action space is a compact, convex subset of a Euclidean space, and each player's payoff function is continuous in all actions and quasi-concave in her own action.[1]) This paper regards the class of symmetric, concave games where each player's action space is one-dimensional. This paper establishes that, for such a game, if there is complementarity between each player's action

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[1]This paper refers to these as concave games following Rosen (1965). (Rosen assumed concavity rather than quasi-concavity, but the latter is sufficient for the present purpose.) The result that any concave game has a pure-strategy equilibrium was established near the dawn of game theory. See Fudenberg and Tirole (1991, Theorem 1.2), who attribute the result to three works published in 1952 separately by Debreu, Glicksberg, and Fan.

and an exogenous parameter, then the minimal and maximal symmetric equilibria are both increasing in that parameter. This result mirrors the well-known comparative statics results of Milgrom and Roberts (1990) and Milgrom and Shannon (1994) for (ordinally) supermodular games.[2]

One application is a symmetric market-entry game. Each of a number of identical firms simultaneously decides whether to enter the market or not. Consider the symmetric mixed-strategy equilibria of that game. Suppose there is a change which makes entry more attractive to each firm, holding the actions of the other firms fixed. Following that change, the minimal and maximal symmetric equilibrium entry probabilities both increase. That is true whether there is strategic complementarity between the firms actions, or strategic substitutability, or neither.

A second application is a symmetric Cournot model. Amir and Lambson (2000) establish conditions under which the aggregate production levels are increasing in the number of firms. Here a similar result is established by means of a shorter and simpler argument.[3]

There is another class of symmetric games with one-dimensional action spaces that is known to be similarly amenable to equilibrium comparative statics analysis. That is the class of symmetric games with quasi-increasing best replies. (See Milgrom and Roberts (1994) and Vives (2000, section 2.3.1).) Games with quasi-increasing best replies are somewhat unusual, whereas concave games are very common.[4]

The well-known comparative statics results for games with strategic complementarities rely on Tarski's fixed-point theorem. The comparative statics result for symmetric games with quasi-increasing best replies relies on Tarski's intersection-point theorem.[5] The comparative statics result established here for symmetric, concave games does not rely on one of Tarski's theorems, but instead it relies on a little-known version of the intermediate value theorem for correspondences established by Chipman and Moore in 1971 (see Moore, 1999).

[2]A result closely related to that of Milgrom and Roberts was developed independently by Matt Sobel. The result was later extended to ordinally supermodular games by Milgrom and Shannon (1994). It holds more generally in games with strategic complementarities (Vives, 2005).

[3]The Cournot game is not supermodular. To establish their result, Amir and Lambson transform the game into a decision problem that is supermodular. Here that transformation is unnecessary, because the Cournot game is concave under common conditions.

[4]Furthermore, there are no simple conditions that imply that a game has quasi-increasing best replies. (Amir and Castro (2013) do provide sufficient conditions, but they are not simple.) In contrast, a concave game is defined by conditions on its primitives that are both simple and commonly encountered.

[5]Milgrom and Roberts (1994) overlook that theorem, and instead reestablish a version of it.

2 Univariate Kakutani correspondences

This section first reviews a standard equilibrium comparative statics result for increasing functions, which is based on Tarski's fixed-point theorem. This section then establishes a parallel comparative statics result for correspondences satisfying the conditions of Kakutani's theorem over a one-dimensional space. In the following section, this result is applied to establish a comparative statics result for symmetric, concave games.

Here is a version of Tarski's famous fixed-point theorem.[6]

Theorem (Tarski (1955)). *Let $(X, \geq)$ be a complete lattice, and $f : X \to X$ an increasing function.*

(a) The function f has a largest fixed point $\overline{x}$ and a smallest fixed point $\underline{x}$, and furthermore

(b) If $f(x) \geq x$, then $\overline{x} \geq x$. And, if $x \geq f(x)$, then $x \geq \underline{x}$.

Here and throughout "increasing" means what some authors mean by "nondecreasing," that is $x_H \geq x_L$ implies $f(x_H) \geq f(x_L)$. Say that $\overline{x}$ and $\underline{x}$ are the extremal fixed points of f.

Part (b) of the theorem may be used to establish a comparative statics result for the extremal fixed points: Consider a family of functions $\{f_t\}_{t \in T}$, where for each value of the parameter t, the function f_t meets the conditions of the previous theorem. Part (a) of the theorem implies that f_t has extremal fixed points: $\overline{x}_t$ and $\underline{x}_t$, for each t. Furthermore, part (b) may be used to establish that if $f_t(x)$ is increasing in t for each x, then the extremal fixed points are increasing in t:

Corollary. *Let $(X, \geq)$ be a complete lattice, T a partially ordered set, and $\{f_t : X \to X\}_{t \in T}$ a family of increasing functions.*

If f_t is increasing in t, then both $\overline{x}_t$ and $\underline{x}_t$ are increasing in t.

Proof. Let H and L be a pair of elements in T such that $H \geq_T L$, where $\geq_T$ is the partial order on the set T. That f_t is increasing in t means $f_H(x) \geq f_L(x)$ for all $x \in X$.

Claim: $\overline{x}_H \geq \overline{x}_L$.

$f_H(\overline{x}_L) \geq f_L(\overline{x}_L) = \overline{x}_L$, because f_t is increasing in t, and $\overline{x}_L$ is a fixed point of f_L.

By part (b) of the previous theorem, $f_H(x) \geq x$ implies $\overline{x}_H \geq x$.

[6] The conclusions of Tarski's Theorem 1 are more broadly as follows: the set of fixed points of f is a complete lattice, that set has a largest element $\overline{x}$ and a smallest element $\underline{x}$, $\overline{x} = \sup\{x \in X : f(x) \geq x\}$, and $\underline{x} = \inf\{x \in X : f(x) \leq x\}$. (See also Vives (2000, Theorem 2.2).) Note that $\overline{x} = \sup\{x \in X : f(x) \geq x\}$ implies that if $f(x) \geq x$, then $\overline{x} \geq x$.

The version of Tarski's theorem in the body of this paper states only those conclusions that are used to establish comparative statics in the following corollary.

Thus $\overline{x}_H \geq \overline{x}_L$ as desired.

The proof that $\underline{x}_H \geq \underline{x}_L$ proceeds similarly: $\underline{x}_H = f_H(\underline{x}_H) \geq f_L(\underline{x}_H)$, therefore $\underline{x}_H \geq \underline{x}_L$ by the previous theorem. □

The preceding argument is fairly standard, see for example Vives (2000, section 2.2). The main mathematical result of the present paper is that a parallel argument holds for correspondences satisfying the conditions of Kakutani's theorem, provided that the underlying space is one-dimensional.

Suppose now that X is a non-empty, compact and convex subset of a Euclidean space $\mathbb{R}^k$. Say that $F : X \rightrightarrows X$ is a *Kakutani correspondence* if the graph of F is closed, and for each x, $F(x)$ is a nonempty, compact and convex set. Kakutani's fixed-point theorem implies that such a correspondence has a fixed point. Additionally, the set of fixed points is compact.

Suppose further that $k = 1$; call this the univariate case. Here X reduces to a nonempty, compact interval. Similarly, if F is a Kakutani correspondence, then $F(x)$ is a nonempty, compact interval for each x. Write $F(x) = [\underline{F}(x), \overline{F}(x)]$. Of course, any compact subset of $\mathbb{R}$ has greatest and least elements. So in the univariate case, a Kakutani correspondence has a greatest fixed point $\overline{x}$ and a least fixed point $\underline{x}$. This is analogous to part (a) in Tarski's theorem above. The main mathematical result of the present paper is analogous to part (b) in Tarski's theorem:

Theorem 1. *Let $X \subset \mathbb{R}$ be a nonempty and compact interval, and $F : X \rightrightarrows X$ a Kakutani correspondence.*

(a) The correspondence F has a greatest fixed point $\overline{x}$ and a least fixed point $\underline{x}$, and furthermore

(b) If $\overline{F}(x) \geq x$, then $\overline{x} \geq x$. And, if $x \geq \underline{F}(x)$, then $x \geq \underline{x}$.

Part (a) follows from Kakutani's theorem as just discussed. For increasing functions, part (a) of Tarski's theorem above implies part (b). The same is not true here. Part (b) here does not follow from Kakutani's theorem, but rather from a little-known version of the intermediate value theorem for correspondences, established by Chipman and Moore in 1971 (see Moore, 1999, Theorem 9.39 and p182). That theorem appears in the appendix to this paper.

Proof of (b). Let $G : X \rightrightarrows \mathbb{R}$ be the interval-valued correspondence $G(x) = [\overline{G}(x), \underline{G}(x)]$ where $\overline{G}(x) = \overline{F}(x) - x$ and $\underline{G}(x) = \underline{F}(x) - x$, for all $x \in X$. Notice $0 \in G(x) \Leftrightarrow x \in F(x)$.

Claim: If $\overline{F}(x) \geq x$ for a particular point $x \in X$, then $\overline{x} \geq x$.

$\overline{F}(x) \geq x$ implies $\overline{G}(x) \geq 0$.

$\underline{G}(\max X) = \underline{F}(\max X) - \max X \leq 0$.

Chipman and Moore's theorem (stated in the appendix) implies that if $Y \subset X$ is an interval, then the image $G(Y)$ is an interval. Thus $G([x, \max X]) \supset [\underline{G}(\max X), \overline{G}(x)] \ni 0$.

So there is a point $y \in [x, \max X]$, where $0 \in G(y)$, so $y \in F(y)$. Recall $\overline{x}$ is the largest fixed point of F, so $\overline{x} \geq y \geq x$, as desired.

The proof of the second part of (b) proceeds similarly: If $x \geq \underline{F}(x)$, then $G([\min X, x]) \supset [\underline{G}(x), \overline{G}(\min X)] \ni 0$. So there is a point $y \in [\min X, x]$ where $y \in F(y)$, which implies $x \geq y \geq \underline{x}$ as desired. □

It is remarkable that in the univariate case, the set of fixed points of a Kakutani correspondence has a similar structure to the set of fixed points of an increasing function (or more generally of an increasing correspondence), that is the structure described in parts (b) of the two theorems in this section. Just as part (b) of Tarski's theorem yields a comparative statics result for the fixed points of a family of increasing functions, part (b) of the theorem here yields a comparative statics result for the fixed points of a family of univariate Kakutani correspondences:

Given a family of such correspondences, $\{F_t\}_{t \in T}$, let $\overline{x}_t$ be the greatest fixed point and $\underline{x}_t$ the least fixed point, of F_t.

Corollary 2. *Let $X \subset \mathbb{R}$ be a nonempty and compact interval, T a partially ordered set, and $\{F_t : X \rightrightarrows X\}_{t \in T}$ a family of Kakutani correspondences.*

If $\overline{F}_t$ and $\underline{F}_t$ are increasing in t, then both $\overline{x}_t$ and $\underline{x}_t$ are increasing in t.

The proof of this corollary mirrors the proof of the previous corollary to Tarski's theorem.

Proof. Let H and L be a pair of elements in T such that $H \geq_T L$.

Claim: $\overline{x}_H \geq \overline{x}_L$.

$\overline{F}_H(\overline{x}_L) \geq \overline{F}_L(\overline{x}_L) \geq \overline{x}_L$, because $\overline{F}_t$ is increasing in t, and $\overline{x}_L$ is a fixed point of F_t.

Part (b) of the previous theorem then implies that $\overline{x}_H \geq \overline{x}_L$

The proof that $\underline{x}_H \geq \underline{x}_L$ proceeds similarly: $\underline{x}_H \geq \underline{F}_H(\underline{x}_H) \geq \underline{F}_L(\underline{x}_H)$, therefore $\underline{x}_H \geq \underline{x}_L$ by the previous theorem. □

The next section applies this corollary to establish a comparative statics result for the class of games considered in this paper.

3 Comparative statics in symmetric, concave games

This section presents the main result, which is an equilibrium comparative statics result for symmetric, concave games with one-dimensional action spaces.

Consider a game in strategic form (Fudenberg and Tirole, 1991). Such a game G consists of three elements: a finite set of players $I = \{1, 2, ..., n\}$, and for each player $i \in I$ an action (or pure-strategy) space A_i, and a payoff function $u_i : A \to \mathbb{R}$, where $A = \times_{i \in I} A_i$.

A *concave game* is one where each player's action space A_i is a compact, convex subset of a Euclidean space $\mathbb{R}^k$, and each player's payoff function u_i is continuous in all actions a and quasi-concave in the player's own action a_i. A concave game with a one-dimensional action space is a concave game where $k = 1$, so each A_i is a compact interval in $\mathbb{R}$.

A *quasi-symmetric game* is one where $A_1 = A_j$ for each player j, and $u_1(x, y, ..., y) = u_2(y, x, y, ..., y) = ... = u_n(y, ..., y, x)$ for each pair of actions x and y in A_1.[7] Notice that a quasi-symmetric game is determined by I, A_1 and u_1.

Given a partially ordered space T, the payoff function $u_i : A \times T \to \mathbb{R}$ satisfies the *single-crossing property* in (a_i, t) if for all $t^H \geq t^L$, $a_i^H \geq a_i^L$, and a_{-i}, $u_i(a_i^H, a_{-i}, t^L) \geq u_i(a_i^L, a_{-i}, t^L)$ implies $u_i(a_i^H, a_{-i}, t^H) \geq u_i(a_i^L, a_{-i}, t^H)$, and $u_i(a_i^H, a_{-i}, t^L) > u_i(a_i^L, a_{-i}, t^L)$ implies $u_i(a_i^H, a_{-i}, t^H) > u_i(a_i^L, a_{-i}, t^H)$.[8]

Theorem 3. *Let $\Gamma_t = \{I, A_1, u_1(a_1, a_{-1}, t)\}_{t \in T}$ be a family of quasi-symmetric, concave games with one-dimensional action spaces. For each t, there is a greatest symmetric equilibrium $\overline{a}(t)$ and a least symmetric equilibrium $\underline{a}(t)$.*

If $u_1(a_1, a_{-1}, t)$ satisfies the single-crossing property in (a_1, t), then $\overline{a}(t)$ and $\underline{a}(t)$ are both increasing in the parameter t.

Proof. Fix a value t in the partially ordered set T, and consider the game G_t.

Consider $B_t : A_1 \rightrightarrows A_1$ where $B_t(y)$ is player one's best-response set when all other players play y and the parameter value is t, that is $B_t(y) = \arg\max_{x \in A_1} u_1(x, y, ..., y, t)$. A symmetric, pure-strategy equilibrium of the game with parameter t is a profile $(x, ..., x) \in A_1^n$ where $x \in B_t(x)$.

Because the game is concave, B_t is a Kakutani correspondence, by the classic argument.[9] Here where A_1 is one-dimensional, B_t is a univariate Kakutani correspondence.

[7]In two-player games, quasi-symmetry is equivalent to the usual definition of symmetry. In games with more than two players, quasi-symmetry is weaker than the usual definition of symmetry. See Reny (1999) and Plan (2017).

[8]See Milgrom and Shannon (1994). As later authors have suggested, the property would more aptly be named "single-crossing differences."

[9]For this argument in the asymmetric case, see the proofs of Theorems 1.1 and 1.2 in Fudenberg and Tirole (1991). For this argument specifically in the symmetric case considered here, see the proof of "Nash's Theorem for Symmetrical Games" in Moulin (1986, 115).

By part (a) of the previous theorem, B_t has extremal fixed points, $\overline{x}_t$ and $\underline{x}_t$, in A_1. The greatest symmetric equilibrium is $\overline{a}(t) = (\overline{x}_t, ..., \overline{x}_t) \in A_1^n$. The least symmetric equilibrium is $\underline{a}(t) = (\underline{x}_t, ..., \underline{x}_t) \in A_1^n$.

Now consider the family of games $\{\Gamma_t\}_{t\in T}$. If u_1 satisfies the single-crossing property in (a_1, t), then $\overline{B}_t$ and $\underline{B}_t$ are increasing in t, by Milgrom and Shannon (1994, Theorem 4). Then by the previous corollary, $\overline{x}_t$ and $\underline{x}_t$ are both increasing in t, as desired. □

4 Applications

This section presents two applications.

4.1 A market-entry game

Consider an n-player quasi-symmetric game with a binary action space $A_1 = \{0, 1\}$. For example, this might be a market-entry game, where the players are identical firms and the action $a_i = 1$ indicates that firm i enters the market while $a_i = 0$ indicates that firm i does not enter. Let $u_1 : A \times T \to \mathbb{R}$ be the payoff function of player one. We will see that if u_1 satisfies the single-crossing property in (a_1, t), then the extremal symmetric equilibria in mixed strategies are increasing in t.

Consider the symmetric equilibria in mixed strategies. Equivalently, consider the symmetric, pure-strategy equilibria in the mixed extension of the game: now let $\alpha_i \in [0, 1]$ denote the probability that firm i enters the market, and $u_1(\alpha; t)$ the expected payoff of player one.

As is well known, the mixed extension of a finite game is a concave game. Here where the underlying space of pure actions is binary, the mixed-action space is one-dimensional. Thus, for each t, the game meets the conditions of the previous section.

Suppose that $u_1(a; t) = \phi_1(a) - c(t)$, where $c(t)$ is the cost of entry, and $\phi_1(a)$ is the payoff of firm 1 gross of the entry cost. If $c(t)$ is decreasing in t, then u_1 satisfies the single-crossing property in (α_1, t). In that case the extremal symmetric equilibria, $\overline{\alpha}_t$ and $\underline{\alpha}_t$, are both increasing in t. Notice that nothing has been assumed about the function ϕ_1. That is, the game may be one of strategic complementarity, strategic substitutability, or neither.

4.2 The effect of entry in a Cournot market

Amir and Lambson (2000) investigate the effects of entry in Cournot markets. They establish conditions where as the number of firms increases, the aggregate production level also increases. That result does not directly follow from the previous theory of games with strategic complementarities. Instead Amir and Lambson extend and adapt that theory to establish the result.[10] Here a similar result is established by means of a shorter and simpler argument.

Consider a symmetric Cournot oligopoly. Each of the t firms simultaneously chooses a production level q_i in the interval $[0, k]$. The profit of firm i is $\pi_i(q) = q_i * p(x) - c(q_i)$, where $x = \sum_{i=1}^{t} q_i$ is the aggregate production level, $p : [0, \infty) \to [0, \infty)$ is the inverse demand function and $c : [0, k] \to [0, \infty)$ is the cost function shared by all firms.

Suppose that the inverse demand function p is decreasing and log-concave, the cost function c is convex, and both are twice differentiable. (Those are common assumptions in a Cournot model.) Then the game is a concave one. It has a greatest and a least symmetric equilibrium in pure strategies. Let $\overline{x}_t$ and $\underline{x}_t$ be the resulting aggregate production levels. We will see that both are increasing in t.

Consider a change of variables where $q_1 = (y/t) + d$. Here y is a candidate aggregate production level to be divided evenly across the firms, and d is the deviation of firm 1. Then the payoff of firm 1 is

$$u_1(d, y, t) = \left(\frac{y}{t} + d\right) * p(y + d) - c\left(\frac{y}{t} + d\right).$$

Let $G_t(y) = \arg\max_d u_1(d, y, t)$ be the set of optimal deviation levels for firm 1. The value x_t is an equilibrium aggregate production level if $0 \in G_t(x_t)$. So $\overline{x}_t$ and $\underline{x}_t$ are the greatest and least roots of G_t.

Notice that $\partial^2 u_1/\partial d\partial t \geq 0$, so G_t is increasing in t.[11] Thus corollary 5, stated in the appendix, implies that the extremal equilibrium aggregate production levels, $\overline{x}_t$ and $\underline{x}_t$, are increasing in the number of firms, t.

Appendix

Theorem (Chipman and Moore)**.** *If $F : S \rightrightarrows T$, where S is connected topological space, T is an arbitrary topological space, F is upper hemicontinuous on S, and if, for each $x \in S$,*

[10] The Cournot game is not supermodular. Amir and Lambson transform player one's decision problem into a supermodular one. Here that is unnecessary because the game is concave.

[11]
$$\frac{\partial u_1}{\partial t} = -\frac{x}{t^2}\left(p(x+d) - c'\left(\frac{x}{t} + d\right)\right)$$
$$\frac{\partial}{\partial d}\frac{\partial u_1}{\partial t} = -\frac{x}{t^2}\left(p'(x+d) - c''\left(\frac{x}{t} + d\right)\right)$$

Both p' and $-c''$ are non-positive, so $\partial^2 u_1/\partial d\partial t \geq 0$ as claimed. Thus u_1 is supermodular in (d, t) by the standard argument for smooth functions. So G_t is increasing in t by Topkis's theorem.

$F(x)$ is a (non-empty) connected subset of T, then $F(S)$ is a connected subset of T.[12]

Notice that a Kakutani correspondence meets the conditions of the theorem. (That a Kakutani correspondence has closed graph implies that it is upper hemicontinuous. Convex-valuedness implies connected-valuedness.)

Here is version of the main theorem of this paper for the roots, rather than the fixed points, of a correspondence. A *root* (or zero) of the correspondence $G : X \rightrightarrows \mathbb{R}$ is a point x where $0 \in G(x)$.

Theorem 4. *Let $X \subset \mathbb{R}$ be a nonempty and compact interval, and $G : X \rightrightarrows \mathbb{R}$ a Kakutani correspondence where $\overline{G}(\min X) \geq 0$ and $\underline{G}(\max X) \leq 0$.*

(a) The correspondence G has a greatest root $\overline{x}$ and a least root point $\underline{x}$, and furthermore

(b) If $\overline{G}(x) \geq 0$, then $\overline{x} \geq x$. And, if $0 \geq \underline{G}(x)$, then $x \geq \underline{x}$.

Proof. (a) That G has some root follows from the theorem of Chipman and Moore. The set of roots is closed because the graph of G is closed. Thus the set of roots is a compact subset of $\mathbb{R}$, so it has a greatest and a least element.

(b) If $\overline{G}(x) \geq 0$, then the restriction of G to the interval $[x, \max X]$ has a root by part (a), so $\overline{x} \geq x$. Similarly if $\underline{G}(x) \leq 0$, then G has a root in the interval $[\min X, x]$, so $x \geq \underline{x}$. □

Corollary 5. *Let $X \subset \mathbb{R}$ be a nonempty and compact interval, T a partially ordered set, and $\{G_t : X \rightrightarrows X\}_{t \in T}$ a family of correspondences which each meet the conditions of the previous theorem.*

If $\overline{G}_t$ and G_t are increasing in t, then both $\overline{x}_t$ and $\underline{x}_t$ are increasing in t.

Proof. Let H and L be a pair of elements in T such that $H \geq_T L$.

Claim: $\overline{x}_H \geq \overline{x}_L$.

$\overline{G}_H(\overline{x}_L) \geq \overline{G}_L(\overline{x}_L) \geq 0$, because $\overline{G}_t$ is increasing in t, and $\overline{x}_L$ is a root of G_L.

Part (b) of the previous theorem then implies that $\overline{x}_H \geq \overline{x}_L$

The proof that $\underline{x}_H \geq \underline{x}_L$ proceeds similarly: $\underline{0}_H \geq \underline{G}_H(\underline{x}_H) \geq \underline{G}_L(\underline{x}_H)$, therefore $\underline{x}_H \geq \underline{x}_L$ by the previous theorem. □

References

Amir, R. and L. D. Castro (2013): "Nash Equilibrium in Games with Quasi-Monotonic Best-Responses," Available at lucianodecastro.net.

[12]This is Theorem 9.39 in Moore (1999, p145). The assumption that S and T are topological spaces is stated on page 118 of that book. The attribution of the theorem to a 1971 work of Chipman and Moore is stated on page 182.

Amir, R. and V. E. Lambson (2000): "On the Effects of Entry in Cournot Markets," *The Review of Economic Studies*, 67, 235–254.

Fudenberg, D. and J. Tirole (1991): *Game Theory*, MIT Press.

Milgrom, P. and J. Roberts (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255–1277.

——— (1994): "Comparing Equilibria," *The American Economic Review*, 84, 441–459.

Milgrom, P. and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

Moore, J. C. (1999): *Mathematical Methods for Economic Theory 2*, Springer-Verlag Berlin Heidelberg.

Moulin, H. (1986): *Game Theory for the Social Sciences*, NYU Press, 2 ed.

Plan, A. (2017): "Symmetric n-player games," Available at asafplan.com.

Reny, P. J. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

Rosen, J. B. (1965): "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," *Econometrica*, 33, 520–534.

Tarski, A. (1955): "A lattice-theoretical fixpoint theorem and its applications." *Pacific J. Math.*, 5, 285–309.

Vives, X. (2000): *Oligopoly Pricing: Old Ideas and New Tools*, MIT.

——— (2005): "Complementarities and Games: New Developments," *Journal of Economic Literature*, 43, 437–479.