

Symmetric n -player games

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Abstract

This paper considers symmetry in games with more than two players. It is often noted that a two-player game is symmetric if it looks the same to both players. However, when there are more than two players, the most common definition of a symmetric game requires more than that the game looks the same to all of its players. Previous authors have established that games which are symmetric in the common sense have a number of useful properties. With few exceptions, those properties continue to hold in the richer class of games that look the same to all players.

Keywords: symmetric games, symmetrical games

1 Introduction

This paper considers symmetry in games with more than two players. For games with just two players, the concept of symmetry is broadly understood. (A two-player game is symmetric if both players share the same strategy space and if the players swap strategies, then they swap payoffs.) For games with more than two players, the matter is not so clear. There are a number of different ways in which the standard definition of a symmetric two-player game may be extended to games with more than two players. A two-player game is symmetric if it looks the same to both players. This paper investigates the class of n -player games that look the same to all players. The common definition of a symmetric n -player game excludes many games in that more general class, including some prominent economic models. Previous authors have established that games which are symmetric in the common sense have a number of useful properties. With few exceptions, those useful properties continue to hold in games that are symmetric in the more general sense that they look the same to all players.

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The following two games are symmetric in different senses. Both games are simplified oligopoly models in which each of the firms simultaneously chooses a price between zero and one. Supposing production is costless, each firm's payoff is equal to its price times its demand.

- In the first model, each firm's demand is equal to one minus the firm's own price, plus the average price of the other firms.
- The second model is a version of Salop's model of price competition in a circular city. Each firm's demand is equal to one minus the firm's own price, plus the average price of the two adjacent firms.

Notice that when there are just two firms, the two models coincide with one another, and with a version of Hotelling's model of price competition in a linear city: each firm's demand is equal to one minus the firm's own price, plus the other firm's price. The two-player game is symmetric in the well known sense: for any pair of prices (x, y) , the payoff of firm 1 when he sets price x and his rival sets price y is the same as the payoff of firm 2 when she sets price x and her rival sets price y . The two models also coincide in the case of three firms.

When there are four or more firms, the two models differ, and they are symmetric in different senses. The first model is symmetric in the common sense. The second model is not symmetric in the common sense, though it clearly possesses some aspects of symmetry.

The first model is *totally symmetric*: every permutation of the players preserves the payoff structure of the game. (Many authors simply refer to games satisfying that condition as symmetric, but this paper follows von Neumann and Morgenstern (1953) in referring to such games as totally symmetric.) Total symmetry has been the most commonly considered concept of symmetry in n -player games.

In the second model, not every player permutation preserves the payoff structure, but some do. In that model, a permutation preserves the payoff structure if it preserves the identity of each firm's neighbors. A rotation of the circle that maps player i to player j , $i + 1$ to $j + 1$, and so on, does preserve the payoff structure of the game. A transposition of any two firms does not preserve the payoff structure, if there are four or more firms in total.

The second model is *weakly symmetric*: for each pair of players i and j , there exists a permutation of all the players that maps player i to player j and preserves the payoff structure of the game. As noted by von Neumann and Morgenstern, the set of player permutations that preserve the payoff structure of a game forms a group, the *symmetry group*. In the language of group theory, a game is weakly symmetric if its symmetry group has a single orbit, which contains the entire set of players.¹ Thus weak symmetry formalizes the natural

¹Fix some player i . Permutations in the symmetry group map i to various other players. The *orbit* of

idea that “a symmetric game is one that looks the same to all the players” (Binmore, 2007).²

A number of works in applied game theory consider models that are weakly symmetric but not totally symmetric. Perhaps the most prominent example is the model of price competition in a circular city popularized by, and often attributed to, Salop (1979). A closely related model was developed earlier by Vickrey in 1964 (reprinted in Vickrey, Anderson, and Braid, 1999). Salop’s model has been extended by later authors, including Grossman and Shapiro (1984) who consider competition in both price and advertisement. The aforementioned models all share the rotational symmetry of a circle. A model may be weakly symmetric without such rotational symmetry: a prominent example is the model of elimination tournaments developed by Rosen (1986, section III). A more recent example is the model of price competition on a network developed by Ushchev and Zenou (2016).³

Symmetry is encountered frequently in pure and applied game theory. A two-player game is symmetric if it looks the same to both players. However, in a game with more than two players, the common definition of symmetry, that is total symmetry, requires more than that the game looks the same to all of its players, that is weak symmetry. Previous authors have established that totally symmetric games have a number of useful properties. This paper shows that, with few exceptions, those useful properties continue to hold in the richer class of weakly symmetric games.

A basic property of totally symmetric games is that if in some symmetric strategy profile one player cannot profitably deviate, then no player can profitably deviate, so the profile is a symmetric equilibrium. A number of authors have used this property to establish conditions under which totally symmetric games have symmetric equilibria. This paper shows that the basic property continues to hold in weakly symmetric games, and consequently the previous results establishing existence of symmetric equilibria in totally symmetric games continue to hold in weakly symmetric games.

A second basic property of totally symmetric games is that they are fair in the sense that for every pair of players i and j , if there is an equilibrium where player i gets some payoff v , then there is another equilibrium where j gets the same payoff v . One implication of this property is that in a totally symmetric, supermodular game, the minimal and maximal equilibria are symmetric (Milgrom and Roberts, 1990); a fact which in turn has multiple useful

player i is the set of players that may be reached in that way. Weak symmetry says that all players may be reached in that way, so the orbit of each player is the entire set of players. This is discussed formally in footnote 10.

²While Binmore states that informal idea, and defines symmetric two-player games, he does not state a formal definition for symmetric games with more than two players.

³They show that if the network is vertex-transitive, then a symmetric equilibrium exists in their model. Their model is weakly symmetric if and only if the network is vertex-transitive. More generally, a network game is weakly symmetric if and only if the network is vertex-transitive.

applications. This paper shows that weakly symmetric games have the same basic fairness property, and consequently the same implication holds for weakly symmetric, supermodular games.

A third basic property of totally symmetric games is that the symmetry-invariance criterion popularized by Harsanyi and Selten (1988) selects only symmetric strategy profiles in such games. Thus not only do symmetric equilibria exist in totally symmetric games, but there is an argument to be made for focusing on those equilibria in such games. This paper shows that weak symmetry is both sufficient and necessary for this third basic property.

Previous authors have established that totally symmetric games have a number of additional properties, which, this paper shows, continue to hold in weakly symmetric games. This paper finds one basic property of totally symmetric games that does not continue to hold in weakly symmetric games:

Total symmetry is related to anonymity. Following Blonski (2000), say that a game is *anonymous* if for each trio of distinct players i , j and k , swapping the strategies of players i and j does not affect the payoff of player k . Total symmetry requires anonymity, whereas weak symmetry does not. In fact, total symmetry is equivalent to the combination of weak symmetry and anonymity, as shown in this paper. As just discussed, in many instances where total symmetry has been assumed, weak symmetry would suffice. Equivalently, in many instances where total symmetry has been assumed, the indirect assumption of anonymity is superfluous. While that is true in many instances, it is not true regarding the following property.

A fourth basic property of totally symmetric games follows. Beginning with an n -player games, fix the strategies of all but two players, and consider the resulting game played by the remaining two players. If the original n -player game is totally symmetric, then the resulting two-player game is symmetric. Relating to that property, Amir (1996) and Hefti (2017) establish conditions under which totally symmetric games have only symmetric equilibria. That basic property does not continue to hold in weakly symmetric n -player games; for that property, the additional assumption of anonymity is necessary. Consequently, weakly symmetric games may have asymmetric equilibria under conditions where totally symmetric games have only symmetric equilibria.

von Neumann and Morgenstern (1953, section 28) give a general discussion of symmetry in n -player games in normal form. Their discussion provides a foundation for the concept of weak symmetry considered in this paper. As they note, “a real understanding of the nature and structure of symmetry is not possible without some familiarity with (at least) the elements of group theory.” While von Neumann and Morgenstern do not explicitly mention weak symmetry, to a reader familiar with their discussion of symmetry in games,

and the theory of permutation groups, the concept of weak symmetry is readily apparent. Recall, the set of player permutations that preserve the payoff structure of a game forms a group, the symmetry group. A game is weakly symmetric if its symmetry group has a single orbit, which contains the entire set of players. Equivalently, a game is weakly symmetric if its symmetry group is transitive, which is a commonly considered condition in group theory. An n -player game is totally symmetric if its symmetry group is n -transitive, which is a stronger condition.

It seems that von Neumann and Morgenstern's discussion of symmetry in n -player games was largely overlooked by later authors who beginning in the 1980s reconsidered that topic. However, there are two papers before this one that briefly consider implications of weak symmetry:

Maynard-Smith defines evolutionarily stable strategies for two-player symmetric games. Palm (1984) extends that definition to arbitrary n -player games. He notes that his definition reduces to a simpler one in the case where the game is totally symmetric. He briefly mentions that his definition reduces in the same way when instead the game satisfies the property of weak symmetry.

Peitz (1999) investigates equilibrium uniqueness in symmetric, supermodular games. He notes that there are versions of Salop's circle model that are not totally symmetric, but do satisfy a weaker concept of symmetry. That weaker concept of symmetry is not made entirely precise, but seems to coincide with the concept of weak symmetry considered in this paper. Peitz argues that a particular property previously established for totally symmetric, supermodular games, continues to hold under his weaker concept of symmetry.⁴

Reny (1999) establishes general conditions for equilibrium existence in discontinuous games. In addition, he establishes conditions for the existence of symmetric equilibria in games that have "enough symmetry." For that purpose, he proposes quasi-symmetry, which is weaker than weak symmetry. He shows that quasi-symmetry may be used to establish the existence of symmetric equilibria in pure strategies. Reny mistakenly suggests that quasi-symmetry may be similarly used to establish the existence of symmetric equilibria in mixed strategies. As this paper shows in section 4, weak symmetry implies quasi-symmetry of the game's mixed extension, which in turn is sufficient for Reny's purpose, and several others.

Section 2 introduces weak symmetry and describes its relationship to total symmetry. Sections 3 and 4 establish implications of weak symmetry. Section 5 describes a further implication of total symmetry. Section 6 concludes.

⁴See the discussion in footnote 15.

2 Weak and total symmetry

This section introduces weak symmetry and describes its relationship to the most common concept of symmetry in n -player games, that is total symmetry. In two-player games, the two concepts of symmetry coincide. More generally, total symmetry is equivalent to the combination of weak symmetry and an additional condition known as anonymity. There are prominent economic models that satisfy weak symmetry but not anonymity, and thus not total symmetry. The following sections show that for many results where total symmetry has previously been assumed, weak symmetry would suffice instead.

Consider a game, in strategic (or normal) form (Fudenberg and Tirole, 1991). Such a game consists of three elements: a finite set of players $I = \{1, 2, \dots, n\}$, and for each player $i \in I$ a pure-strategy space S_i , and a payoff function $u_i : S \rightarrow \mathbb{R}$, where $S = \times_{i \in I} S_i$. When considering mixed strategies, suppose that S is finite, and that each player seeks to maximize her expected payoff.

A two-player game is symmetric if $S_1 = S_2$ and for all pairs of strategies $(x, y) \in S_1^2$, $u_1(x, y) = u_2(y, x)$.⁵ This paper regards the generalization of that standard definition of symmetry to games with more than two players. Every concept of symmetry considered here requires that the players share a common strategy space, but the different concepts of symmetry impose different restrictions on the profile of payoff functions.

A permutation π on the set of players is a bijection $\pi : I \rightarrow I$. Following von Neumann and Morgenstern (1953, section 28), say that a game is *symmetric* with respect to a permutation π if $S_i = S_{\pi(i)}$ for all players i , and the permutation preserves the payoff structure of the game in the following sense,

$$u_{\pi(i)}(s_1, \dots, s_n) = u_i(s_{\pi(1)}, \dots, s_{\pi(n)}) \text{ for all } s \in S \text{ and } i \in I.$$

⁵von Neumann and Morgenstern (1953) state that definition in the case where the game is zero-sum. The extension to the general-sum case is straightforward, and is given, for example, by van Damme (1991) in bimatrix games. Osborne and Rubinstein (1994, Exercise 20.4) state the analogous definition for two-player, ordinal games (ordinal games are defined in terms of the players preferences rather than their payoff functions).

This is the standard concept of symmetry in two-player games, but there exist other, more general, concepts of symmetry for such games, see for example Cao and Yang (2016) and the references therein. The present paper could begin with a more general concept of symmetry in two-player games and then consider further generalization of that concept from two-player games to n -player games. In that case, instead of considering von Neumann and Morgenstern's concept of the symmetries of a game, one would consider Nash's (1951) more general concept of its symmetries, or Harsanyi and Selten's (1988) still more general concept of the automorphisms of a game. Beginning instead with this standard definition of a symmetric, two-player game, and von Neumann and Morgenstern's concept of the symmetries of a game, substantially simplifies the exposition of this paper.

If a game is symmetric with respect to π , say that π is a *symmetry* of the game. As von Neumann and Morgenstern emphasize, the set of such symmetries forms a group, the *symmetry group*.⁶ In particular, if π is a symmetry of the game, then so is its inverse π^{-1} , and if in addition ξ is a symmetry of the game, then so is the composition $\pi \circ \xi$.

As von Neumann and Morgenstern remark, a game may be symmetric with respect to some but not all of the player permutations. The common definition of a symmetric, n -player game is at the upper extreme:

Definition. A game is *totally symmetric* if it is symmetric with respect to every permutation π on I .

Following Dasgupta and Maskin (1986), many authors simply refer to such games as “symmetric games.” Following instead von Neumann and Morgenstern (1953) and Palm (1984), this paper specifically refers to such games as “totally symmetric games.” (Separately, note that Dasgupta and Maskin’s often-repeated definition of a symmetric game is not quite correct.⁷)

This paper investigates the following more general concept of symmetry in n -player games.

Definition. A game is *weakly symmetric* if for each pair of players i and j , the game is symmetric with respect to some permutation π on I where $\pi(i) = j$.

Equivalently, a game is weakly symmetric if for each player i , the game is symmetric with respect to some permutation π where $\pi(i) = i$.⁸

⁶In general, a symmetry of some structured object is a transformation of that object that preserves its structure. The set of such symmetries of an object, here a game, form a group, referred to as the symmetry group. A group is a set of elements, here permutations of I , together with an operation, here composition, that satisfies the four group axioms: Here (1) the group of symmetries is closed under inversion, (2) the group of symmetries is closed under composition, (3) the identity map is a symmetry, and (4) the composition of symmetries is associative.

For games, the transformations considered here, and by von Neumann and Morgenstern, are permutations and the symmetry group is specifically a permutation group. Dixon and Mortimer (1996) is a standard reference on permutation groups.

⁷They say that a game is symmetric if $u_i(s) = u_{\pi(i)}(s_{\pi(1)}, \dots, s_{\pi(n)})$ for all $s \in S$, $i \in S$ and permutations $\pi : I \rightarrow I$. Notice that here u_i is on the left and $u_{\pi(i)}$ is on the right. von Neumann and Morgenstern (1953, p256 fn4) explicitly caution that this statement of symmetry is incorrect. (Given their lengthy discussion of the issue, it seems likely that Morgenstern or von Neumann had earlier made the error that Dasgupta and Maskin later repeat, but I do not find written evidence of that. von Neumann and Morgenstern give the correct statement of symmetry in the original 1944 edition.) In two-player games, Dasgupta and Maskin’s definition of symmetry is equivalent to the standard one. However, in games with three or more players, their definition implies that the game is not only totally symmetric, but it is also a game of common interest, so $u_i(s) = u_j(s)$ for all i, j and s . That has been established independently by Vester (2012) and Ham (2013). From the discussion in Dasgupta and Maskin’s companion paper, it is clear that they did not intend that a symmetric game is one of common interest.

⁸Clearly the general definition implies the special case where $j = i$. To see the converse, suppose that

In two-player games, weak and total symmetry coincide with one another and the standard definition of a symmetric, two-player game.⁹ In games with three or more players, total symmetry implies weak symmetry, but not the converse. Total symmetry requires that the game is symmetric with respect to every permutation while weak symmetry requires only that it is symmetric with respect to a sufficiently broad subset of permutations.

In the language of group theory: A game is weakly symmetric if its symmetry group is transitive. Equivalently, a game is weakly symmetric if its symmetry group has a single orbit, which contains the entire set of players.¹⁰ Thus weak symmetry formalizes the natural idea that “a symmetric game is one that looks the same to all the players” (Binmore, 2007).

Example (Circular prisoner’s dilemma). The symmetric, two-player prisoner’s dilemma may be extended to an n -player game that is totally symmetric. It may also be extended to a different n -player game that is weakly symmetric but not totally symmetric.

Begin with the following two-player prisoner’s dilemma. Each of the two players simultaneously chooses an action $a_i \in \{0, 1\}$. The payoff of player i is $-a_i + 2a_j$, where j denotes the other player.

The two-player game may be extended to a totally symmetric, n -player game: Each player’s strategy space is $\{0, 1\}$ as before. The payoff of player i is now $-a_i$ plus two times the average of the other players’ actions, that is $u_i(a) = -a_i + \frac{2}{n-1} \sum_{j \neq i} a_j$.

The two-player game may also be extended to a weakly symmetric, n -player game: the “circular prisoner’s dilemma.” Each player’s strategy space is $\{0, 1\}$ as before. Let $\rho : I \rightarrow I$ be the permutation where $\rho(1) = 2$, $\rho(2) = 3$, ..., $\rho(n-1) = n$, $\rho(n) = 1$. This permutation ρ is called a *rotation* (or a *circular shift*). The payoff of player i is now $u_i(a) = -a_i + 2a_{\rho(i)}$. In the case where $n = 3$, the payoff functions are as follows.

$$u_1(a) = -a_1 + 2a_2$$

$$u_2(a) = -a_2 + 2a_3$$

$$u_3(a) = -a_3 + 2a_1$$

the game is symmetric with π and ρ where $\pi(i) = 1$ and $\xi(j) = 1$. Then the game is symmetric with respect to the permutation $\xi^{-1} \circ \pi$. Note $\xi^{-1}(\pi(i)) = j$ as desired.

⁹When $n = 2$, there are only two permutations on I : the identity which maps player 1 to 1 and 2 to 2, and the transposition which maps 1 to 2 and 2 to 1. Every game is symmetric with respect to the identity. In two-player games, both weak and total symmetry simply require that the game is also symmetric with respect to the transposition of 1 and 2.

¹⁰Let Γ denote the group of symmetries of the game. That Γ is *transitive* means that for each i and j there exists $\pi \in \Gamma$ such that $\pi(i) = j$. (Transitivity is the condition stated in my definition of a weakly symmetric game.) The *orbit* of player i is the set of players $\{j \in I : \exists \pi \in \Gamma \text{ such that } j = \pi(i)\}$. Notice Γ is transitive if and only if the orbit of each player is the entire player set I . (Dixon and Mortimer (1996) is an excellent text on the theory of permutation groups. They describe transitivity and orbits as here on their page 8.)

In a game with three players, there are six permutations of the player set. Three of those permutations are transpositions of two players. This game is not symmetric with respect to any of the three transpositions, so it is not totally symmetric.¹¹ A fourth permutation is the identity map on I , which is a symmetry of every game. The last two permutations are the aforementioned rotation ρ and the double rotation $\rho \circ \rho$. This game is symmetric with respect to the rotation ρ , so it is also symmetric with respect to the composition of ρ with itself, that is the double rotation. The single rotation maps player 3 to 1. The double rotation maps player 2 to 1. Thus the game is weakly symmetric.

The prisoner's dilemma may be interpreted as a game of pollution externalities, where the action 1 means pollute and the action 0 means abate. The two-player prisoner's dilemma may be interpreted as such a game between two nations. The circular prisoner's dilemma may then be interpreted as a game of airborne pollution externalities between a number of nations located around the equator, where the wind blows from East to West, so that each nation is affected by the pollution emitted by its neighbor to the East.

The circular prisoner's dilemma is *rotationally symmetric*: it is symmetric with respect to the aforementioned rotation ρ .

Rotational symmetry implies weak symmetry, but not the converse. Recall the famous model of price competition in a circular city popularized by Salop (1979). That model is rotationally symmetric and thus weakly symmetric. The same is true in related models including Grossman and Shapiro (1984). With four or more firms, these circular-city models are not totally symmetric. (Unlike the circular prisoner's dilemma, Salop's circle model also has reflectional symmetry: it is symmetric with respect to the reflection π where $\pi(i) = n + 1 - i$.) A game may be weakly symmetric without being rotationally symmetric, as in the following example, which is a simplified version of the model of elimination tournaments developed by Rosen (1986).

Example (A four-player, two-stage elimination tournament). Consider first a symmetric, two-player Tullock contest. Each of the two players, 1 and 2, simultaneously chooses an effort level $a_i \in [0, \infty)$. One of the two players wins the contest and receives a prize of value v . The probability that player 1 wins is $a_1/(a_1 + a_2)$. The payoff of the winning player i is $v - a_i$, while the payoff of the losing player j is $-a_j$. This is a symmetric, two-player game.

Now consider a four-player game where players 1 and 2 engage in the previous Tullock contest, and players 3 and 4 simultaneously engage in a separate but identical contest. This game is weakly symmetric but not totally symmetric. (It is not symmetric with respect to

¹¹For example, consider the transposition of players 1 and 2: $\tau(1) = 2$, $\tau(2) = 1$, and $\tau(3) = 3$, the strategy profile $(0, 0, 1)$, and the player $i = 1$. Notice $u_{\tau(1)}(s) = u_2(0, 0, 1) = 2 \neq 0 = u_1(0, 0, 1) = u_1(s_{\tau(1)}, \dots, s_{\tau(n)})$.

the transposition of players 1 and 3. It is symmetric with respect to the transposition of players 1 and 2, and with respect to the transposition of players 3 and 4. It is also symmetric with respect to the permutation that interchanges the pair (1, 2) with the pair (3, 4), that is the permutation π where $\pi(1) = 3$, $\pi(2) = 4$, $\pi(3) = 1$ and $\pi(4) = 2$.)

This may be extended to a four-player, two-stage game, which is an elimination tournament. The first stage is the same as the four-player game above, but the two winners in the first stage, instead of immediately receiving a prize of value v , move on to the second stage. The second stage is the two-player Tullock contest that was initially considered. The two losers in the first stage have no role in the second stage. Again this is a weakly symmetric game, which has the same symmetry group as above.

This elimination tournament is a simplified version of the weakly symmetry tournament that Rosen considers in his section III. (In that section, he considers more generally a weakly symmetric tournament with n stages and 2^n homogeneous contestants. In his section IV, he considers a version with heterogeneous contestants, which is not symmetric.)

Total symmetry is related to the anonymity condition considered by Blonski (2000). A game has a *common strategy space* if $S_1 = S_j$ for each player j .

Definition. A game with a common strategy space is *anonymous* if for each player i , the following is true. For all permutations π on I where $\pi(i) = i$,

$$u_i(s) = u_i(s_{\pi(1)}, \dots, s_{\pi(n)}) \text{ for all } s \in S.$$

Equivalently, a game with a common strategy space is anonymous if for all distinct trios of players i, j and k , and strategy profiles s , swapping the strategies of players i and j does not affect the payoff of player k . For example, Cournot's model of competition in quantities is anonymous. The definition of anonymity may be extended to games without a common strategy space.¹²

Theorem 1. *Total symmetry is equivalent to the combination of weak symmetry and anonymity.*¹³

Proof. That total symmetry implies weak symmetry is clear. Similarly, that total symmetry implies anonymity. Here I prove that weak symmetry and anonymity imply total symmetry.

¹²A game is anonymous if for each player i , the following is true. For all permutations π such that $\pi(i) = i$, $u_i(s) = u_i(s_{\pi(1)}, \dots, s_{\pi(n)})$ for all $s \in S$ such that $(s_{\pi(1)}, \dots, s_{\pi(n)}) \in S$. Every two-player game trivially meets this condition, whether or not the game has a common strategy space. (That is true because in a two-player game, for any fixed i , the only permutation where $\pi(i) = i$ is the identity.)

¹³Ham (2013, Theorem 3.15) similarly asserts that “full symmetry” is equivalent to the combination of “standard symmetry” and “weak anonymity.” However, his proof that standard symmetry plus weak anonymity implies full symmetry is incorrect. Following correspondence between him and me, Ham has independently developed a revised proof of his theorem, which may appear in a future version of his paper.

To establish total symmetry, I must show that for every s, i and π , $u_{\pi(i)}(s) = u_i(s_{\pi(1)}, \dots, s_{\pi(n)})$. Fix some s, i and π .

(1) Let $j = \pi(i)$. Weak symmetry implies that there exists some permutation ξ where $\xi(i) = j = \pi(i)$ such that

$$u_{\pi(i)}(s) = u_{\xi(i)}(s) = u_i(s_{\xi(1)}, \dots, s_{\xi(n)}).$$

(2) Anonymity implies that for any permutation α such that $\alpha(\xi(i)) = \xi(i)$,

$$u_i(s_{\xi(1)}, \dots, s_{\xi(n)}) = u_i(s_{\alpha(\xi(1))}, \dots, s_{\alpha(\xi(n))}).$$

(Notice that on the left hand side, player i is playing $s_{\xi(i)}$. On the right hand she is playing $s_{\alpha(\xi(i))}$, which is the same as on the left.)

(3) Now set $\alpha = \pi \circ \xi^{-1}$, so $\alpha(\xi(j)) = \pi(j)$ for all j . Note that $\alpha(\xi(i)) = \pi(i) = \xi(i)$, as was assumed in the previous step (2).

Combining the previous three steps yields $u_{\pi(i)}(s) = u_i(s_{\pi(1)}, \dots, s_{\pi(n)})$ as desired. \square

Every two-player game is trivially anonymous. So the previous proposition implies that in two-player games, total and weak symmetry are equivalent.

In games with three or more players, total symmetry requires anonymity, whereas weak symmetry does not. Consequently, total symmetry restricts u_1 , whereas weak symmetry does not, in the following sense. In a two-player game, symmetry does not restrict u_1 : for any function $\phi : S_1^2 \rightarrow \mathbb{R}$ there exists a symmetric two-player game where $u_1 = \phi$. Similarly, in an n -player game, weak symmetry does not restrict u_1 : for any $\phi : S_1^n \rightarrow \mathbb{R}$ there exists a weakly symmetric, n -player game where $u_1 = \phi$.¹⁴ On the other hand, total symmetry does restrict u_1 if $n \geq 3$: it requires that u_1 satisfies the condition of anonymity. For example, total symmetry requires that swapping the strategies of players 2 and 3, does not affect the value of u_1 .

A number of previous authors have established various properties of totally symmetric games. Many of those properties continue to hold in the richer class of weakly symmetric games, as shown in the next two sections.

¹⁴For any such ϕ , there exists a rotationally symmetric game where $u_1(s) = \phi(s)$, $u_2(s) = \phi(s_2, s_3, \dots, s_n, s_1)$, $u_3(s) = \phi(s_3, s_4, \dots, s_n, s_1, s_2)$ and so on. Recall rotational symmetry implies weak symmetry.

3 Implications of weak symmetry

This section describes some implications of weak symmetry, and of symmetries in games more generally. Weakly symmetric games are fair in a sense, which has further implications for supermodular games. In weakly symmetric games, the equilibrium selection criterion known as symmetry-invariance selects only symmetric equilibrium. Furthermore, weak symmetry is not only sufficient for that property, but is also in a sense necessary.

It is well known that in a totally symmetric game, if s^* is an equilibrium, then every permutation of s^* is also an equilibrium, and if v^* is an equilibrium payoff profile, then every permutation of v^* is also an equilibrium payoff profile. A related property holds for weakly symmetry games. Both properties follow from the following more fundamental result regarding games with symmetries.

In every game, the set of equilibria is symmetry-invariant, and the set of equilibrium payoff profiles is similarly symmetry-invariant, in the following sense.

Theorem 2. *Consider an n -player game that is symmetric with respect to some player permutation π .*

- (a) *If $s^* \in S$ is an equilibrium, then $(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*)$ is also an equilibrium.*
- (b) *If $v^* \in \mathbb{R}^n$ is an equilibrium payoff profile, then $(v_{\pi(1)}^*, \dots, v_{\pi(n)}^*)$ is also an equilibrium payoff profile.*

The theorem is stated for equilibria in pure strategies, but notice that it immediately extends to equilibria in mixed strategies, because if π is a symmetry of the game, then π is also a symmetry of the game's mixed extension.

Proof. (a) That s^* is an equilibrium means that for each player i , $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$.

To show that $(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*)$ is also an equilibrium, it suffices to show that for each player j , and for each $s_j \in S_j$, $u_j(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*) \geq u_j(s_j, s_{\pi(-j)}^*)$, where $(s_j, s_{\pi(-j)}^*)$ denotes the strategy profile formed by replacing player j 's strategy in the profile $(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*)$ with the alternative strategy s_j .

Let $i = \pi(j)$. Noting that then $S_i = S_j$, let $s_i = s_j$. Then

$$u_j(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*) = u_i(s^*) \geq u_i(s_i, s_{-i}^*) = u_j(s_j, s_{\pi(-j)}^*)$$

as desired. Here the two equalities are true because π is a symmetry and $i = \pi(j)$, and the inequality is true because s_i^* is a best response for player i to s^* .

(b) Suppose v^* is an equilibrium payoff profile, so there exists an equilibrium s^* such that $u(s^*) = v^*$. Part (a) established that $(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*)$ is also an equilibrium. Notice $u_i(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*) = u_{\pi(i)}(s^*) = v_{\pi(i)}^*$ by symmetry. So $u(s_{\pi(1)}^*, \dots, s_{\pi(n)}^*) = (v_{\pi(1)}^*, \dots, v_{\pi(n)}^*)$ as desired. \square

A totally symmetric game is symmetric with respect to every player permutation. In that case the previous theorem yields the aforementioned property that if s^* is an equilibrium, so is every permutation of s^* . In a weakly symmetric game, it instead yields the following related property.

Corollary 3. *Consider a weakly symmetric game. Let i and j be any pair of players.*

(a) *If s^* is an equilibrium, then there exists another equilibrium s^{**} where $s_j^{**} = s_i^*$, and s^{**} is a permutation of s^* .*

(b) *If v^* is an equilibrium payoff profile, then there exists another equilibrium payoff profile v^{**} where $v_j^{**} = v_i^*$, and v^{**} is a permutation of v^* .*

Thus weakly symmetric games, like totally symmetric games, are fair in the sense that if there is an equilibrium where player i gets payoff μ then there is also an equilibrium where player j gets payoff μ .

Proof. Fix some pair of players i and j . That the game is weakly symmetric means that it is symmetric with respect to some permutation π where $\pi(i) = j$.

(a) Let $s^{**} = (s_{\pi(1)}^*, \dots, s_{\pi(n)}^*)$. The previous theorem established that s^{**} is an equilibrium. Note $s_j^{**} = s_{\pi(j)}^* = s_i^*$, as desired.

(b) Suppose $v^* = u(s^*)$. Let $v^{**} = u(s^{**})$. Again $v_j^{**} = v_i^*$ as desired. \square

The fairness property has additional implications in supermodular games. Topkis showed that in any supermodular game, the set of equilibria (in pure strategies) has a largest element and a smallest element. That is, such a game has extremal equilibria \bar{s} and \underline{s} , where $\bar{s} \geq s^* \geq \underline{s}$ for each equilibrium s^* . Milgrom and Roberts (1990) observe that if a supermodular game is totally symmetric, then its extremal equilibria are both symmetric. So if a totally symmetric, supermodular game has only one symmetric equilibrium, then that is the unique equilibrium, symmetric or otherwise.¹⁵ The same remains true more generally in totally

¹⁵Milgrom and Roberts further establish that in a supermodular game, the extremal equilibria bound each players serially undominated strategies. It follows that if a totally symmetric, supermodular game has a unique symmetric equilibrium in pure strategies, then the game is dominance solvable. (They state that as the third corollary after their Theorem 5.)

Peitz argues that the aforementioned property holds for a more general concept of symmetry, which he refers to as “local symmetry.” Peitz (1999, Proposition 2) states “Any locally symmetric, ordinal supermodular game with a single symmetric equilibrium candidate has a unique pure strategy Nash equilibrium and

symmetric games with strategic complementarities (Vives, 2005). The present paper shows that the same remains true replacing the assumption of total symmetry with the more general assumption of weak symmetry:

Corollary 4. *If a game with strategic complementarities is weakly symmetric, then its extremal equilibria \bar{s} and \underline{s} are both symmetric.*

Again, it follows that if a weakly symmetric game with strategic complementarities has just one symmetric equilibrium, then that is the unique equilibrium.

Proof. Recall that a game with strategic complementarities, such as a supermodular game, has extremal equilibria.

Claim: If a weakly symmetric game has a largest equilibrium \bar{s} , then $\bar{s}_i = \bar{s}_j$ for each pair of players i and j .

Because \bar{s} is an equilibrium, by the previous corollary, there exists another equilibrium s^{**} such that $s_j^{**} = \bar{s}_i$. Then $\bar{s} \geq s^{**}$ implies $\bar{s}_j \geq s_j^{**} = \bar{s}_i$. Considering the pair (j, i) rather than the pair (i, j) similarly implies $\bar{s}_i \geq \bar{s}_j$. Thus $\bar{s}_i = \bar{s}_j$, as desired.

The argument for the smallest equilibrium \underline{s} is similar. □

In totally symmetric games, there is a common argument for focusing on symmetric equilibria. That argument holds equally in weakly symmetric games. As Kreps (1990) suggests, “if the game is symmetric... both we and the players involved might find asymmetric equilibria somewhat unintuitive.” That suggestion reflects an equilibrium selection criterion, symmetry-invariance, which was introduced by Nash (1951) and extended and popularized by Harsanyi and Selten (1988). Symmetry-invariance has the same implications for weakly symmetric games as for totally symmetric games.

A strategy profile σ is *symmetric* if $\sigma_i = \sigma_j$ for all players i and j . A strategy profile σ is *symmetry-invariant* if for each symmetry π of the game, $(\sigma_1, \dots, \sigma_n) = (\sigma_{\pi(1)}, \dots, \sigma_{\pi(n)})$.¹⁶

is dominance solvable.” Defining local symmetry, Peitz writes an oligopoly “game is called *locally symmetric* if for any pair of firms’ indices (a, b) there exists a permutation of firms’ indices such that for all prices the profits of any firm i become the profits of some firm j after permutation and in particular firm a ’s profits become firm b ’s profits.”

It seems that what Peitz has in mind is weak symmetry, though his definition is not entirely precise. However, in the proof of his proposition 2, Peitz makes a claim that is not quite true under the assumption of weak symmetry, but would be true given a more restrictive concept of symmetry. (That is the claim “... Any such permutation is an equilibrium.”)

¹⁶The present paper, like von Neumann and Morgenstern, considers transformations of the game that rearrange the players. Nash considered transformations that both rearrange the players and the strategy spaces, and Harsanyi and Selten considered transformations that additionally rescale the payoff functions. What this paper calls a symmetry of the game, Nash would also call a symmetry of the game, but by Nash’s definition there may be additional symmetries. However, typically, weakly or totally symmetric games do not have symmetries in Nash’s sense that are not symmetries in the sense considered here.

Of course, if a strategy profile is symmetric, then it is symmetry-invariant. It is known that the converse is true in totally symmetric games. Palm (1984) remarks that it remains true in weakly symmetric games. In fact, weak symmetry is not only sufficient but also necessary for that important property:

Theorem 5. *Consider a game that has at least two distinct strategy profiles. The set of symmetric strategy profiles coincides with the set of symmetry-invariant strategy profiles if and only if the game is weakly symmetric.*

Proof. In any game, symmetry of σ implies symmetry-invariance of σ .

Claim: If the game is weakly symmetric, then symmetry-invariance implies symmetry.

Consider a weakly symmetric game. Fix any pair of players i and j . There exists a symmetry π such that $\pi(i) = j$. If σ is symmetry-invariant, then $\sigma_i = \sigma_{\pi(i)} = \sigma_j$, as desired.

Claim: If the game is not weakly symmetric and it has two distinct strategy profiles, then there exists a strategy profile that is symmetry-invariant but not symmetric.

Let Γ denote the group of symmetries of the game. Let $O(i) = \{\pi(i) : \pi \in \Gamma\}$ be the orbit of player i . The collection of orbits $\{O(i) : i \in I\}$ forms a partition of the set of players I . A game is weakly symmetric if and only if there is a single orbit. All players who share an orbit, share the same strategy space. A strategy profile is symmetry-invariant if all players who share the same orbit, play the same strategy.

That the game is not weakly symmetric means that it has at least two distinct orbits. Suppose the game has $m \geq 2$ orbits. Choose a representative player from each orbit, that is choose m players i_1, \dots, i_m such that $\{O(i_1), \dots, O(i_m)\}$ is a partition of I . For each representative player $k = \{1, \dots, m\}$ choose a strategy $s_{i_k} \in S_{i_k}$. Form the symmetry-invariant strategy profile where $s_i = s_{i_k}$ if $i \in O(i_k)$. If it is not symmetric then we are done. Suppose instead it is symmetric. That the game has two distinct strategy profiles means that there is some player j that has at least two distinct strategies, so j has some strategy that is distinct from the current strategy s_j . Replace s_j with that other strategy, and similarly replace the strategy of every player in j 's orbit with that other strategy. The new strategy profile remains symmetry-invariant, but is no longer symmetric. \square

Nash showed the following.

Theorem (Nash (1951, Theorem 2)). *Any finite game has a symmetry-invariant equilibrium in mixed strategies.*¹⁷

¹⁷Nash's Theorem 2 states verbatim "Any finite game has a symmetric equilibrium point." A point of potential confusion is that Nash when writes "symmetric" with respect to a strategy profile, he means what is now meant by "symmetry-invariant." Of course the meanings of these two coincide in symmetric games. (Varian writing in 1980 is an early example where, regarding a strategy profile, "symmetric" is used in the modern sense rather than the sense of Nash.)

A corollary to Nash’s theorem is that every finite, totally symmetric game has a symmetric equilibrium in mixed strategies. It seems that this corollary was known to Nash, but he did not explicitly state it.¹⁸ That result was later independently proven by Dasgupta and Maskin (1986, Lemma 6).¹⁹ In fact, the corollary remains true under weak symmetry:

Corollary 6. *Any finite, weakly symmetric game has a symmetric equilibrium in mixed strategies.*

That is true because in a weakly symmetric game, just as in a totally symmetric game, every symmetry-invariant strategy profile is symmetric.

This section has established two main results for games with symmetries, and applied them to weakly symmetric games. The next section shows that weak symmetry implies an alternative condition, quasi-symmetry, and establishes implications of quasi-symmetry.

4 Implications of quasi-symmetry

This section considers an alternative condition, quasi-symmetry, proposed by Reny (1999). Weak symmetry implies quasi-symmetry. Reny observes that quasi-symmetry may be used in place of total symmetry to establish the existence of symmetric equilibria in pure strategies. This section shows that quasi-symmetry, and thus weak symmetry, suffice in place of total symmetry for several purposes.

A well known property of totally symmetric games is that if in some symmetric strategy profile one player cannot profitably deviate, then no player can profitably deviate, so the profile is a symmetric equilibrium. A number of authors have used this property to establish various conditions under which totally symmetric games have symmetric equilibria.

As Reny (1999) suggests, for pure strategies, the aforementioned property follows from a condition weaker than total symmetry:

¹⁸Nash’s 1950 dissertation largely coincides with the paper published in 1951. In the acknowledgements of both the paper and the dissertation, Nash mentions, “David Gale suggested the investigation of symmetric games.” In the introduction to the dissertation, Nash mentions “The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern... the existence of a symmetrical equilibrium point in a symmetrical game.” However, in the body of the paper and the dissertation, there is no definition of a symmetric(al) game, nor any explicit discussion at all of such games. His only formal result regarding symmetry is his Theorem 2. One presumes that Nash and Gale were familiar with von Neumann and Morgenstern’s definition of a totally symmetric game.

The corollary is stated for symmetric, two-player bimatrix games by van Damme (1991, Lemma 9.2.1).

¹⁹Regarding that result, Dasgupta and Maskin remark, “This result is well known among game theorists. We are including a proof here because we have been unable to find a reference.” It seems that they, and some of their contemporaries, overlooked the connection to Nash’s theorem 2.

Definition. A game is *quasi-symmetric* if for all players i and j , $S_i = S_j$, and

$$u_1(x, y, \dots, y) = u_2(y, x, y, \dots, y) = \dots = u_N(y, \dots, y, x) \text{ for all } x, y \in S_1.$$

In two-player games, quasi-symmetry is equivalent to symmetry. In games with three or more players, weak symmetry implies quasi-symmetry, but not the converse.

von Neumann and Morgenstern say that a game is *totally unsymmetric* if the game is not symmetric with respect to any permutation apart from the identity. A quasi-symmetric game with more than two players may be totally unsymmetric.²⁰ (Thus the prefix “quasi” is apt.) Consequently, the results of the previous section regarding games with symmetries need not have any bite for quasi-symmetric games. Nonetheless, quasi-symmetry does suffice in place of total symmetry for several purposes.

In an arbitrary game, let $B_i : S \rightarrow S_i$ denote player i 's best-reply correspondence,

$$B_i(s) = \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}).$$

In a game with a common strategy space, consider the restriction of the best-reply correspondence to symmetric strategy profiles: $B_i^d(s_1) = B_i(s_1, \dots, s_1)$. Say that $B_i^d : S_1 \rightarrow S_1$ is player i 's *diagonal best-reply correspondence*.

In a quasi-symmetric game, all players share the same diagonal best-reply correspondence, that is $B_1^d = B_j^d$ for all players $j \in I$. Consequently,

Lemma 7. *In a quasi-symmetric game, if $s_1^* \in B_1^d(s_1^*)$, then $(s_1^*, \dots, s_1^*) \in S_1^n$ is a symmetric equilibrium.*

This result is implicit in Reny's proof of his Theorem 4.1, though he does not explicitly state it. Of course, because total symmetry implies weak symmetry, which implies quasi-symmetry, this result remains true in totally or weakly symmetric games.

Proof. In a quasi-symmetric game, $s_1^* \in B_1^d(s_1^*)$ implies $s_1^* \in B_j^d(s_1^*)$ for each player j , which implies $(s_1^*, \dots, s_1^*) \in B(s_1^*, \dots, s_1^*)$, which is to say that the symmetric strategy profile (s_1^*, \dots, s_1^*) is an equilibrium. \square

A number of authors considering totally symmetric games have established conditions under which there exists a fixed point of the diagonal best-reply correspondence B_1^d , and

²⁰Begin with a totally symmetric game with more than three players, such as the totally symmetric extension of the prisoner's dilemma considered before. Now modify the payoff function as follows. If at least $n - 1$ players are playing the same pure strategy, then let the payoffs be as in the original game. For all other strategy profiles, let the payoff of player i be equal to i . If $n \geq 3$, then the resulting game has no symmetries apart from the identity.

consequently there exists a symmetric equilibrium in pure strategies by the previous argument. For example, Moulin (1986, p115) shows that in totally symmetric, concave games, and Vives (1990) shows that in totally symmetric, supermodular games. Because the previous result holds under quasi-symmetry, the existence results of Moulin and Vives may be extended from totally symmetric games to quasi-symmetric games.²¹

The argument in the previous lemma may be extended to mixed strategies. Let Σ_1 denote the set of mixed strategies for player 1, and let $B_1^d : \Sigma_1 \rightarrow \Sigma_1$ now denote the diagonal best-reply correspondence for mixed strategies. In a totally symmetric game, if σ_1^* is a fixed point of B_1^d , then $(\sigma_1^*, \dots, \sigma_1^*)$ is a symmetric equilibrium in mixed strategies. Reny (1999, Corollary 5.3) seems to suggest that this argument continues to hold replacing total symmetry with quasi-symmetry again, but that is not quite correct. Instead, it holds under the assumption that the mixed extension of the game is quasi-symmetric, that is:

Definition. A game is *mixed-quasi-symmetric* if for all players i and j , $S_i = S_j$, and

$$u_1(x, \psi, \dots, \psi) = u_2(\psi, x, \psi, \dots, \psi) = \dots = u_N(\psi, \dots, \psi, x) \text{ for all } x \in S_1 \text{ and } \psi \in \Sigma_1.$$

Note that this definition simply replaces the pure strategy y in the previous definition with the mixed strategy ψ . Mixed-quasi-symmetry is equivalent to quasi-symmetry of the game's mixed extension.

The previous lemma may be extended as follows. In a mixed-quasi-symmetric game, $B_1^d(\psi) = B_j^d(\psi)$ for each $\psi \in \Sigma_1$ and $j \in I$. Consequently,

Corollary 8. *In a mixed-quasi-symmetric game, if $\sigma_1^* \in B_1^d(\sigma_1^*)$, then $(\sigma_1^*, \dots, \sigma_1^*) \in \Sigma_1^n$ is a symmetric equilibrium.*

Thus mixed-quasi-symmetry, or weak symmetry, may be used in place of total symmetry to establish the existence of symmetric equilibria in mixed strategies. For example,

Theorem 9. *Any finite, mixed-quasi-symmetric game has a symmetric equilibrium in mixed strategies.*

²¹A *concave game* is one where each S_i is a compact, convex Euclidean subspace, and each u_i is continuous in s and concave, or quasi-concave, in s_i . In such a game, B_1^d has a fixed point s_1^* by Kakutani's theorem.

In a supermodular game, B_1^d has an increasing selection. That increasing selection has a fixed point s_1^* by Tarski's theorem.

In both cases, if the game is totally symmetric, then (s_1^*, \dots, s_1^*) is a symmetric equilibrium. The same is true if the game is quasi-symmetric rather than totally symmetric, by the previous lemma.

Reny (1999, Theorem 4.1) shows the existence of symmetric pure strategy equilibria in games that are quasi-symmetric, compact, diagonally quasi-concave and diagonally better-reply secure.

This theorem is a slight generalization of corollary 6. Previous authors have shown that in a finite game with a common strategy space, there exists a mixed strategy that is a fixed point of player one's diagonal best-reply correspondence. The theorem then follows by the previous corollary.

Proof. Claim: In a finite game with a common strategy space, there exists a mixed strategy σ_1^* that is a fixed point of player one's diagonal best-response correspondence, that is $\sigma_1^* \in B_1^d(\sigma_1^*)$.

This claim has been proven by Dasgupta and Maskin (1986), via Fan's lemma. Alternatively, note that B_1^d meets the conditions of Kakutani's theorem: Fudenberg and Tirole (1991, Theorem 1.1) prove more generally that Σ is a compact, convex, nonempty subset of a Euclidean space; and B is nonempty-valued, convex-valued, and upper hemicontinuous. Σ_1 and B_1^d inherit those same properties, so Kakutani's theorem implies that B_1^d has a fixed point σ_1^* .

By the previous corollary, if the game is mixed-quasi-symmetric, then $(\sigma_1^*, \dots, \sigma_1^*)$ is a symmetric equilibrium. □

Mixed-quasi-symmetry is a condition on the payoff functions for mixed strategies. It is desirable to have instead a condition on the payoff functions for pure strategies that implies mixed-quasi-symmetry. Weak symmetry is such a condition:

Theorem 10. *Weak symmetry implies mixed-quasi-symmetry, which implies quasi-symmetry.*

The converses are not generally true, quasi-symmetry does not imply mixed-quasi-symmetry, which does not imply weak symmetry.²² In two-player games, the three conditions are equivalent.

²²A network game (Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv, 2010) on a regular graph is mixed-quasi-symmetric, but need not be weakly symmetric. It is weakly symmetric if the graph is vertex-transitive. Here I construct a three-player game that is mixed-quasi-symmetric but not weakly symmetric:

Begin with the following rotationally symmetric three-player game: $S_1 = \{0, 1\}$, $u_1(s) = s_1 + s_2 - s_3$, $u_2(s) = s_2 + s_3 - s_1$, $u_3(s) = s_3 + s_1 - s_2$. Let $\{i, j, k\}$ be any permutation of the player set $\{1, 2, 3\}$. Notice that if $\sigma_j = \sigma_k$, then the expected payoff of player i is equal to $E[\sigma_i]$. That is true for all i , so the game is mixed-quasi-symmetric.

Now consider a modification of the game where player 1's payoff function is replaced with $\tilde{u}_1(s) = s_1 + 2s_2 - 2s_3$. Notice that if $\sigma_2 = \sigma_3$, then $\tilde{u}_1(\sigma_1, \sigma_2, \sigma_2) = E[\sigma_1] + 2E[\sigma_2] - 2E[\sigma_2] = E[\sigma_1]$, so the modified game is still mixed-quasi-symmetric. However, it is no longer weakly symmetric: there is no symmetry of the game that maps player 2 or player 3 to player 1.

It remains unclear what condition on the payoff functions for pure strategies is equivalent to mixed-quasi-symmetry.

Proof. That mixed-quasi-symmetry implies quasi-symmetry is immediate, because for each pure strategy y there exists a mixed strategy ψ^y that places probability one on y , and consequently ψ^y is payoff equivalent to y .

Here I show that weak symmetry implies quasi-symmetry. Considering players 1 and 2, weak symmetry implies that there exists a player permutation π where $\pi(2) = 1$, and

$$u_{\pi(2)}(\sigma) = u_2(\sigma_{\pi(1)}, \dots, \sigma_{\pi(2)}).$$

Setting $\sigma = (x, \psi, \dots, \psi)$, and noting that $\pi(2) = 1$, and $\pi(j) \neq 1$ for all other players $j \neq 2$, the previous line becomes

$$u_1(x, \psi, \dots, \psi) = u_2(\psi, x, \psi, \dots, \psi)$$

as desired. One may similarly show that $u_2(\psi, x, \psi, \dots, \psi) = \dots = u_N(\psi, \dots, \psi, x)$. \square

Dasgupta and Maskin (1986) establish conditions under which equilibria exist in discontinuous, asymmetric games. They seek to further establish conditions under which there exists a mixed strategy equilibrium that is atomless on the discontinuity set. They establish that is true in totally symmetric games that meet their other conditions. All three of their results for totally symmetric games remain true for mixed-quasi-symmetric games.

Quasi-symmetry, and thus weak symmetry, may be used in place of total symmetry for purposes other than establishing the existence of symmetric equilibria:

Milgrom and Roberts (1994) show that equilibrium comparative statics may be established in symmetric games under weaker conditions than in asymmetric games. If the game is symmetric and S_1 is one-dimensional, then the problem of finding symmetric equilibria in pure strategies reduces to the problem of finding fixed points of the map $B_1^d : S_1 \rightarrow S_1$. That is a one-dimensional fixed point problem. In an asymmetric game, even if each S_i is one-dimensional, the problem of finding equilibria is an n -dimensional, fixed-point problem. In the general problem, monotonicity of B may be used to establish equilibrium comparative statics. Milgrom and Roberts show that in the one-dimensional fixed-point problem where S_1 is a compact interval, a weaker condition, quasi-monotonicity of B_1^d , may be similarly used to establish equilibrium comparative statics.²³ They note their result may be applied to totally symmetric games. It may similarly be applied to quasi-symmetric games. Relatedly, Plan (2017) shows that in quasi-symmetric games, concavity suffices in place of strategic

²³When B_1^d is single-valued, it is quasi-increasing if it is “continuous but for upward jumps.” Overlooked by Milgrom and Roberts, Tarski’s intersection point theorem implies the existence of a fixed point for a quasi-increasing function on a compact interval.

quasi-complementarity for the purpose of establishing equilibrium comparative statics.

Hefti (2017, Theorem 1) establishes conditions under which a totally symmetric game has only one symmetric equilibrium. To do so, Hefti considers the diagonal best-response correspondence B_1^d , and establishes conditions, relating to the Index theorem, under which B_1^d has a single fixed-point. That approach is valid not only in totally symmetric games, but more generally in quasi-symmetric games.²⁴ Consequently, Hefti’s Theorem 1 continues to hold replacing the assumption of total symmetry with that of quasi-symmetry.

In general, it is difficult to determine the set of equilibria in an infinitely repeated game. Cronshaw and Luenberger (1994) “restrict attention to games with n identical agents.” They show that for such games the set of strongly symmetric equilibria of the infinitely repeated game has a particularly simple structure. They seem to have in mind the assumption that the stage game is totally symmetric. However, all of Cronshaw and Luenberger’s mathematical statements continue to hold assuming only that the stage game is quasi-symmetric.

This section and the previous one have established that weak symmetry suffices in place of total symmetry for various purposes. That is, in many instances where total symmetry has been invoked, the implied assumption of anonymity is superfluous. The next section describes a result for which both weak symmetry and anonymity are necessary.

5 A further implication of total symmetry

This section describes a property of totally symmetric games that does not hold in weakly symmetric games. Beginning with an n -player game, fix the strategies of all but two players and consider the resulting game played by the remaining two players. If the original n -player game is totally symmetric, then the resulting two-player game is symmetric. Related to this property, Amir (1996), Amir, Jakubczyk, and Knauff (2008), and Hefti (2017) have established conditions under which totally symmetric games have only symmetric equilibria. Those results do not extend to weakly symmetric games.

Consider an arbitrary n -player game G . Fix one player j , and some strategy of that player, s_j . Consider the resulting game played by the remaining $n - 1$ players. That is, consider a new game $\tilde{G}(j, s_j)$ which has a set of $n - 1$ players $\tilde{I} = I \setminus j$, strategy space $\tilde{S} = \times_{i \neq j} S_i$, and payoff functions $\tilde{u}_i : \tilde{S} \rightarrow \mathbb{R}$ where $\tilde{u}_i(\tilde{s}) = u_i(\tilde{s}_1, \dots, \tilde{s}_{j-1}, s_j, \tilde{s}_{j+1}, \dots, \tilde{s}_n)$.

²⁴Hefti refers to that approach as the “Symmetric Opponents Form Approach.” He notes that a number of other authors have taken this approach in studying symmetric equilibria of symmetric games, including Salop.

Proposition 11. *If the n -player game G is totally symmetric, then fixing any player j and strategy s_j , the resulting $(n - 1)$ -player game $\tilde{G}(j, s_j)$ is totally symmetric.*

The same is not true for weakly symmetric games.²⁵

Having fixed one player's strategy, we may now fix a second player's strategy, yielding an $(n - 2)$ -player game which is again totally symmetric. Repeating this procedure, fixing all but two players strategies, yields a symmetric, two-player game. Consequently, conditions under which two-player, symmetric games have only symmetric equilibria may be extended to conditions under which n -player, totally symmetric games have only symmetric equilibria.

Suppose that the n -player game G has an asymmetric equilibria s^* . For some pair of players i and j , $s_i^* \neq s_j^*$. Thus (s_i^*, s_j^*) is an asymmetric equilibrium of the resulting two-player game where the strategy of every other player $k \notin \{i, j\}$ is fixed at s_k^* , as described above. If the original game G is a totally symmetric, strictly supermodular game, then the resulting two-player game is symmetric and strictly supermodular. If a two-player game is symmetric, strictly supermodular, and has one-dimensional strategy spaces, then it has only symmetric equilibria.²⁶ Consequently, totally symmetric, strictly supermodular games with one-dimensional strategy spaces do not have asymmetric equilibria. This result was originally proven by Amir (1996, p145), and later generalized by Amir, Jakubczyk, and Knauff (2008). This result does not extend to weakly symmetric, strictly supermodular games.²⁷ Hefti (2017, Theorem 2) establishes other conditions under which two-player games do not have asymmetric equilibria, and consequently n -player totally symmetric games do

²⁵For example, in the circular prisoner's dilemma with $n \geq 3$, fix $a_n = 1$. In the resulting game with players $\{1, 2, \dots, n - 1\}$, player $n - 1$ has a strategic advantage over the other $n - 2$ players.

²⁶Given one-dimensional strategy spaces, if $s_1 \neq s_2$, then either $s_1 > s_2$ or $s_1 < s_2$. Suppose $s_1 > s_2$. If the game is symmetric and strictly supermodular, then every element of $B_2(s_1)$ is greater than or equal to every element of $B_1(s_2)$. Thus, if $s_1 > s_2$, it cannot be that $s_2 \in B_2(s_1)$ and $s_1 \in B_1(s_2)$. The case where $s_1 < s_2$ is similar. This remains true replacing the assumption of strict supermodularity with the more general assumption that the game has strict strategic complementarities.

²⁷Peitz (1999) remarks that it does not extend to "locally symmetric" games. Local symmetry seems to coincide with weak symmetry. See footnote 15.

Here I show that a four-player, rotationally symmetric, strictly supermodular game may have an asymmetric equilibria. Consider the strategy profile $(1, 3, 2, 4)$. Suppose that the best response correspondence is single-valued and let $b = B_1$. For the profile to be an equilibrium of the rotationally symmetric game, it must be that

$$\begin{aligned} 1 &= b(3, 2, 4) \\ 3 &= b(2, 4, 1) \\ 2 &= b(4, 1, 3) \\ 4 &= b(1, 3, 2). \end{aligned}$$

It is possible to construct a strictly increasing function b where that is true, because none of the four trios of strategies that appear on the right hand side are ranked above another.

It remains unclear whether Amir's result extends to weakly symmetric games with just three players.

not have asymmetric equilibria. Again, this result does not extend to weakly symmetric games.

This paper suggests that for many purposes, other than those described in the current section, the assumption of total symmetry is overly strong. That is, many of the implications of total symmetry continue to hold under weak symmetry. On the other hand, there is a sense in which total symmetry is not strong enough. There is an implication of symmetry in two-player games that does not similarly follow from total symmetry in n -player games: It is well known that in a symmetric, zero-sum, two-player game, each player gets payoff zero in equilibrium. However, in a totally symmetric, zero-sum, n -player game, a player may get a payoff other than zero in equilibrium.²⁸

6 Conclusion

In pure game theory, total symmetry has come to be the most commonly considered concept of symmetry in n -player games. It seems that may have occurred due to a series of historical accidents across the first four published sources that consider symmetry in general n -player games: von Neumann and Morgenstern (1953, originally published in 1944), Nash (1951), Palm (1984) and Dasgupta and Maskin (1986).²⁹ The present paper shows that many results that have previously been established for totally symmetric games continue to hold in weakly symmetric games. Previous work in applied game theory includes prominent models that lie outside the class of totally symmetric games but inside the richer class of weakly symmetric games — perhaps that will also be true in future work.

²⁸If such a game is finite, then there is an equilibrium where each of the n -players gets payoff zero, because a symmetric equilibrium exists. (In a zero-sum game, any symmetric equilibrium yields payoffs of zero.) However, there may be other equilibrium payoff profiles as well. For example, consider the three-player game where each player simultaneously chooses left or right; and if all three players choose the same action, they each get payoff zero, while if only two players choose the same action, they both get payoff $-1/2$ and the third player gets payoff 1. That is a totally symmetric, zero-sum game. One equilibrium is (left, right, right), which yields payoffs $(1, -1/2, -1/2)$.

von Neumann and Morgenstern (1953) did not consider (Nash) equilibrium, but they similarly noted regarding their solutions concepts, “It must be remembered, however, that the concept of fairness and similarly that of total symmetry of the game may or may not imply that all individual players can expect the same fate in an individual play (provided that they play well). For $n = 2$ this implication did hold, but not for $n \geq 3$!” (p259).

Perhaps there is some more restrictive extension of symmetry from two-player games to n -player games, that does preserve the zero-payoff implication in zero-sum games.

²⁹In particular, Dasgupta and Maskin’s famous paper seems to have shaped the later discussion of symmetry in games. von Neumann and Morgenstern’s group-theoretic discussion of symmetry has been largely overlooked. Had Dasgupta and Maskin been aware of the relevant portions of the three earlier works, it seems likely that they would have given a different treatment of symmetry. Weak symmetry was noted by Palm, and would be sufficient in place of total symmetry for all of Dasgupta and Maskin’s purposes.

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